Fixed point structures on a set-mapping pair and cartesian product

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Dedicated to Professor Emeritus Mihail Megan on the occasion of his 75th anniversary

Abstract In this paper we study the following problem (Problem 4.2 in, I.A. Rus, Sets with structure, mappings and fixed point property: fixed point structures, Fixed Point Theory 23, No. 2 (2022), 689-706): Let \((\mathcal{U}, S, M)\) be a fixed point structure on \((\mathcal{U}, M)\). We suppose that \((\mathcal{U}, M)\) is with cartesian product. In which conditions, we have that:

\[ X, Y \in S \Rightarrow X \times Y \in S ? \]

Some results in terms of exponential with fixed point property are also given. The basic results are illustrated by examples.

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1 Introduction

Let \((\mathcal{U}, M)\) be a set-mapping pair (see [29]) and \((\mathcal{U}, S, M)\) be a fixed point structure (f.p.s.) on \((\mathcal{U}, M)\). We suppose that \((\mathcal{U}, M)\) is with cartesian product. In this paper we study the following problem:

In which conditions we have the following implication

\[ X, Y \in S \implies X \times Y \in S ? \]

The structure of the paper is as follows:

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2. Preliminaries

Throughout this paper we follow the notations and terminology from [29]. See also [28], [36], [26].

2 Preliminaries

Following [29] (see also [13]) in what follows we present the basic notions in set-mapping pair theory.

Let \( U \) be a class of nonempty sets with structure and for each ordered pair \((X, Y)\), \(X, Y \in U\), a set of mappings from \(X\) to \(Y\), \(M(X, Y)\) is given. By definition we call the pair \((U, M)\) a set-mapping pair.

We remark that if \((U, M)\) is a set-mapping pair we have the following sets of mappings

\[
M(X, Y) = \{ f : X \to Y \mid f \text{ a mapping} \},
\]

\[
Hom(X, Y) = \{ f : X \to Y \mid f \text{ a morphism} \},
\]

and \(M(X, Y)\).

By \(f \in M\) we understand that there exist \(X, Y\) in \(U\) such that \(f \in M(X, Y)\).

By definition a set-mapping pair \((U, M)\) is with:

- **composition** if \(f, g \in M\) and \(f \circ g\) is defined then \(f \circ g \in M\);
- **identity** if it is with composition and for each \(X \in U\), \(1_X \in M(X, X)\);
- **restriction** if \(U, X, Y \in U\), \(U \subset X\) and \(f \in M(X, Y)\) implies that \(f \mid_U \in M(U, Y)\).

A notion less restrictive as isomorphism (see [3], [18], [28]) is \((U, M)\)-bijection ([29], [13]). A bijective mapping \(f \in M(X, Y)\) is a \((U, M)\)-bijection if for \(h \in M(Y, Y)\) and \(g \in M(X, X)\) we have that \(f^{-1} \circ h \circ f \in M(X, X)\) and \(f \circ h \circ f^{-1} \in M(Y, Y)\).

Another basic notion is that of **retract**. Let \((U, M)\) be a set-mapping pair. By definition, \(Y \in U\) is a retract of \(X \in U\) if there exist two mappings \(r \in M(X, Y)\) and \(s \in M(Y, X)\) such that:

- \(r \circ s = 1_Y\);
- for all \(h \in M(Y, Y)\) and \(g \in M(X, X)\) we have that \(s \circ h \circ r \in M(X, X)\) and \(r \circ g \circ s \in M(Y, Y)\).
We call \( r \) a retraction mapping and \( s \) a coretraction mapping or \((r, s)\) is a retraction-coretraction pair. If \( Y \subseteq X \), then in general we take \( s \) the inclusion mapping.

For retraction theory in various structure sets see: [3], [18], [26], [6], [11], [12], [15], [30], [23], [27], [5], [8], [38].

For some examples of the above notions see [29].

The last notion in this section is that of fixed point structure (f.p.s.) on a set-mapping pair.

Let \((U, M)\) be a set-mapping pair and \( S \subseteq U \), \( S \neq \emptyset \). The triple \((U, S, M)\) is a f.p.s. on \((U, M)\) if for each \( X \in S \) and \( f \in M(X, X) \) we have that the fixed point set of \( f \), \( F_f \neq \emptyset \).

Let \( S_{\text{max}} := \) the class of all \( X \in U \) such that, \( f \in M(X, X) \Rightarrow F_f \neq \emptyset \). By definition the triple \((U, S_{\text{max}}, M)\) is the maximal f.p.s. on \((U, M)\).

For the case in which \( U \subseteq P(U) \), where \( U \) is a set with structure, see [26]. See also [35], [36], [37], [7], [30].

For the fixed point property of sets with structure (ordered sets, topological spaces, metric spaces, Banach spaces, ...) see [1], [3], [7], [11], [15], [18], [20], [25], [26], [30], [36], [5], [16], [8].

3 Set-mapping pairs with cartesian product

The aim of this section is to continue the study in [29] on cartesian product in a set-mapping pair.

By definition a class \( \mathcal{U} \) of sets with structure is with cartesian product if for all \( X, Y \in \mathcal{U} \), the cartesian product of \( X \) and \( Y \), \( X \times Y \), endowed with the usual structure induced by those of \( X \) and \( Y \), \( X \times Y \in \mathcal{U} \).

A set-mapping pair, \((\mathcal{U}, M)\) is with cartesian product if it is with composition, with restriction, \( \mathcal{U} \) is with cartesian product and for each \( f = (f_1, f_2) \in M(X \times Y, X \times Y) \) we have that:

- \( f_1 \in M(X \times Y, X) \) and \( f_2 \in M(X \times Y, Y) \);
- \( f_1(\cdot, y) \in M(X, X) \) and \( f_1(x, \cdot) \in M(Y, X) \), \( \forall x \in X \), \( \forall y \in Y \);
- \( f_2(x, \cdot) \in M(Y, Y) \) and \( f_2(\cdot, y) \in M(X, Y) \), \( \forall x \in X \), \( \forall y \in Y \).

Now we consider the following problems. Let \((\mathcal{U}, M)\) be with cartesian product and \((\mathcal{U}, S, M)\) be a f.p.s. on \((\mathcal{U}, M)\).

**Problem 3.1.** If \( X, Y \in S \) in which conditions \( X \times Y \in S \)?

**Problem 3.2.** If \( X, Y \in S \) in which conditions \( X \times Y \in S_{\text{max}} \)?

**Problem 3.3.** If \( X, Y \in S_{\text{max}} \) in which conditions \( X \times Y \in S_{\text{max}} \)?

Here are some examples with results for these problems.
Example 3.1. Let $\mathcal{U} :=$ the class of all ordered sets, $M(X,Y) := \{f : X \to Y \mid f$ is increasing and $S :=$ the class of all complete ordered sets. If we consider on $X \times Y$ the standard ordered structure (i.e. $(x_1,y_1) \leq (x_2,y_2) \iff x_1 \leq x_2,$ $y_1 \leq y_2$) then $(\mathcal{U}, M)$ is a f.p.s. on $(\mathcal{U}, M)$ and $X, Y$ are in $\mathcal{S}$ then $X \times Y \in \mathcal{S}$. Indeed, if $X$ and $Y$ are complete ordered sets then $X \times Y$ is a complete ordered set. It is well known that, in general, $\mathcal{S} \neq \mathcal{S}_{\max}$ (see references in [29]). It is an open problem whether, if $X, Y \in \mathcal{S}_{\max}$ then $X \times Y \in \mathcal{S}_{\max}$.

Example 3.2. If $(X,d_X)$ and $(Y,d_Y)$ are two metric spaces then we consider on $X \times Y$ the following metrics:

   (1) $d_0_{X\times Y}((x_1,y_1),(x_2,y_2)) := \max \{d_X(x_1,x_2),d_Y(y_1,y_2)\}$;
   
   (2) $d_1_{X\times Y}((x_1,y_1),(x_2,y_2)) := d_X(x_1,x_2) + d_Y(y_1,y_2)$;
   
   (3) $d_2_{X\times Y}((x_1,y_1),(x_2,y_2)) := (d_X^2(x_1,x_2) + d_Y^2(y_1,y_2))^\frac{1}{2}$.

Let us denote by $d_{X\times Y}$ one of these metrics. Now, let $\mathcal{U} :=$ the class of all nonempty metric spaces, $M(X,Y) := \{f : X \to Y \mid f$ is a contraction and $\mathcal{S} :=$ the class of complete metric spaces. If we take on $X \times Y$ the metric $d_{X \times Y}$, then $(\mathcal{U}, M)$ is with cartesian product. This follows from the fact that if $f = (f_1,f_2) : (X \times Y,d_{X \times Y}) \to (X \times Y,d_{X \times Y})$ is a contraction then $f_1 : (X \times Y,d_X) \to (X,d_X)$, $f_2 : (X \times Y,d_X) \to (Y,d_Y)$, $f_1(x,\cdot) : (X,d_X) \to (X,d_X)$, $f_1(x,\cdot) : (Y,d_Y) \to (X,d_X)$, $f_2(\cdot,y) : (X,d_X) \to (Y,d_Y)$ are increasing for all $x \in X$, $y \in Y$.

By contraction principle $(\mathcal{U}, S, M)$ is a f.p.s. on $(\mathcal{U}, M)$.

In this example, if $X, Y$ are in $\mathcal{S}$ then $X \times Y \in \mathcal{S}$. Indeed, if $(X,d_X)$ and $(Y,d_Y)$ are complete metric spaces then $(X \times Y,d_{X \times Y})$ is a complete metric space. In this case, $\mathcal{S} \neq \mathcal{S}_{\max}$ and it is an open problem whether, if $X, Y \in \mathcal{S}_{\max} \implies X \times Y \in \mathcal{S}_{\max}$.

Example 3.3. If $(X,\langle \cdot, \cdot \rangle_X)$ and $(Y,\langle \cdot, \cdot \rangle_Y)$ are two real pre-Hilbert spaces then, on cartesian product, $X \times Y$, we consider the scalar product $\langle \cdot, \cdot \rangle_{X \times Y}$, defined by

$$\langle (x_1,y_1),(x_2,y_2) \rangle_{X \times Y} := \langle x_1,x_2 \rangle_X + \langle y_1,y_2 \rangle_Y.$$ 

With this scalar product, $X \times Y$ is a real pre-Hilbert space.

Let $\mathcal{U} :=$ the class of all real pre-Hilbert spaces and $M(X,Y) := \{f : X \to Y \mid f$ is nonexpansive with $f(X)$ a bounded subset of $Y\}$ and $\mathcal{S} :=$ the class of all real Hilbert spaces. If $X$ and $Y$ are in $\mathcal{U}$ then $X \times Y$ with the scalar product $\langle \cdot, \cdot \rangle_{X \times Y}$ is in $\mathcal{U}$. Moreover $(\mathcal{U}, M)$ is with cartesian product.

By the Browder fixed point theorem (see [15], [30]), $(\mathcal{U}, \mathcal{S}, M)$ is a f.p.s. on $(\mathcal{U}, M)$. In this example, $X, Y \in \mathcal{S} \implies X \times Y \in \mathcal{S}$.

In the fixed point theory on cartesian product the following notion is fundamental.

Definition 3.1. ([29], [37], [39]) Let $(\mathcal{U}, M)$ be a set-mapping pair with cartesian product and $(\mathcal{U}, \mathcal{S}, M)$ be a f.p.s. on $(\mathcal{U}, M)$. By definition, $X \in \mathcal{S}$ is with fixed point selection.
property with respect to $Y \in \mathcal{U}$, on the right, if for all $f \in M(\{x, y\}; X)$ the multivalued operator $P : Y \to X$ defined by $y \mapsto F_{f(y)}$ has a selection $p \in M(Y, X)$. By definition, $Y \in \mathcal{S}$ is with fixed point selection property with respect to $X \in \mathcal{U}$, on the left, if for all $f \in M(\{x, y\}; Y)$ the multivalued operator $Q : X \to Y$ defined by $x \mapsto F_{f(x)}$ has a selection $q \in M(X, Y)$.

In terms of this notion we have the following result.

**Theorem 3.1.** Let $(\mathcal{U}, M)$ be a set-mapping pair with cartesian product and $(\mathcal{U}, \mathcal{S}, M)$ be a f.p.s. on $(\mathcal{U}, M)$. We suppose that for all $X, Y \in \mathcal{S}$ we have:

(1) $X$ is with fixed point selection property with respect to $Y$, on the right;

(2) $Y$ is with fixed point selection property with respect to $X$, on the left.

Then we have the following implication:

$$X, Y \in \mathcal{S} \implies X \times Y \in \mathcal{S}.$$ 

**Proof.** Let $X, Y \in \mathcal{S}$, $f \in M(\{x, y\}; X \times Y)$, $f = (f_1, f_2)$. Since $(\mathcal{U}, M)$ is a set-mapping pair with cartesian product then $f_1 \in M(\{x\}; X)$, $f_2 \in M(\{y\}; Y)$ and from (1) and (2) we have that the multivalued operator $P : Y \to X$ defined by $y \mapsto F_{f(y)}$ has a selection $p \in M(X, Y)$ and the multivalued operator $Q : X \to Y$ defined by $x \mapsto F_{f(x)}$ has a selection $q \in M(X, Y)$.

Since $(\mathcal{U}, M)$ is with composition, the mapping, $q \circ p \in M(Y, Y)$. From $Y \in \mathcal{S}$, there exists $\bar{y} \in Y$ such that, $q(p(\bar{y})) = \bar{y}$. Let us denote, $\bar{x} := p(\bar{y})$. We remark that $(\bar{x}, \bar{y}) \in F_f$, so, $X \times Y \in \mathcal{S}$. \hfill $\square$

## 4 Set-mapping pairs with exponential

We start this section with the following notion.

**Definition 4.1.** ([29]) Let $(\mathcal{U}, M)$ be a set-mapping pair with composition and with restriction. By definition, $(\mathcal{U}, M)$ is with exponential if for all $X, Y \in \mathcal{U}$, $Y^X := M(X, Y)$ endowed with the usual structure induced by those $X$ and $Y$, $M(X, Y) \in \mathcal{U}$.

In the term of this notion we have the following result.

**Theorem 4.1.** Let $(\mathcal{U}, M)$ be a set-mapping pair with cartesian product and with exponential. Let $(\mathcal{U}, \mathcal{S}, M)$ be a f.p.s. on $(\mathcal{U}, M)$. If for $X, Y \in \mathcal{U}$ we have that $M(Y, X) \in \mathcal{S}$, then $X \in \mathcal{S}$ and $X$ has the fixed point selection property with respect to $Y$, to the right and to the left.

**Proof.** First we prove that $X \in \mathcal{S}$. Let $f \in M(X, X)$. Then the mapping $T_f : M(Y, X) \to M(Y, X)$, defined by $h \mapsto f(h)$ is in $M(M(Y, X), M(Y, X))$. This implies that there exists $h^* \in M(Y, X)$ such that $h^* = f(h^*)$, i.e. $F_f \neq \emptyset$.

Now, let us prove, for example, that $X$ has the fixed point selection property with respect to $Y$, to the right. Let $T \in M(X \times Y, X)$. This mapping induces the mapping
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\[ \hat{T} : M(Y, X) \to M(Y, X) \text{ defined by } h \mapsto T(h(\cdot), \cdot). \]
Since \((\mathbb{U}, M)\) is with cartesian product and with exponential then \(\hat{T} \in M(M(Y, X), M(Y, X))\). Since \(M(Y, X) \in \mathcal{S}\), hence that there exists \(h^* \in M(Y, X)\) such that \(h^*(y) = T(h^*(y), y)\), for all \(y \in Y\), i.e., \(X\) has the fixed point selection property with respect to \(Y\), to the right. \(\square\)

**Remark 4.1.** For the case \(\mathcal{S} := \mathcal{S}_{\text{max}}\) we have a result given in [29].

From Theorem 3.1 and Theorem 4.1 we have the basic result of this section.

**Theorem 4.2.** Let the set-mapping pair \((\mathbb{U}, M)\) be with cartesian product and with exponential. Let \((\mathbb{U}, \mathcal{S}, M)\) be a f.p.s. on \((\mathbb{U}, M)\). If for \(X, Y \in \mathbb{U}\) we have that \(M(X, Y) \in \mathcal{S}\) and \(M(Y, X) \in \mathcal{S}\) then \(X \times Y \in \mathcal{S}\).

5 Some problems

5.1

Let \((\mathbb{U}, M)\) be a set-mapping pair with cartesian product and with exponential and \((\mathbb{U}, \mathcal{S}, M)\) be a f.p.s. on \((\mathbb{U}, M)\). Let us consider the following statements with respect to \(X\) and \(Y\) in \(\mathbb{U}\):

- \((S_1)\) \(X\) and \(Y\) are in \(\mathcal{S}\);
- \((S_2)\) \(X \times Y \in \mathcal{S}\);
- \((S_3)\) \(X\) has the fixed point selection property with respect to \(Y\), to the right. and \(Y\) has the fixed point selection property with respect to \(X\), to the left.
- \((S_4)\) \(M(X, Y) \in \mathcal{S}\) and \(M(Y, X) \in \mathcal{S}\).

From the results in [29] and this paper we have that:

- \((i_1)\) \((S_2) \implies (S_1)\);
- \((i_2)\) \((S_3) \implies (S_2)\);
- \((i_3)\) \((S_4) \implies (S_3)\);

These facts give rise to:

**Problem 5.1.** In which conditions \((S_1) \implies (S_2)\)?

**Problem 5.2.** In which conditions \((S_2) \implies (S_3)\)?

**Problem 5.3.** In which conditions \((S_3) \implies (S_4)\)?

References: [3], [18], [26], [39], [36], [40], [34], [33], [10], [32].
5.2

Let \((U_i, M_i), \ i = 1, 2,\) be two set-mapping pairs and \( (U_i, S_i, M_i), \ i = 1, 2,\) be a f.p.s. on \((U_i, M_i), \ i = 1, 2,\) and \(X_1 \times X_2\) the cartesian product of the sets \(X_1\) and \(X_2.\)

For a mapping \(f : X_1 \times X_2 \rightarrow X_1 \times X_2, \ f = (f_1, f_2),\) we suppose that:

1. \(f (\cdot, x_2) \in M_1 (X_1, X_1)\) for all \(x_2 \in X_2;\)

2. we define the multivalued mapping \(P : X_2 \rightarrow X_1\) defined by \(P (x_2) := F_{f_1 (\cdot, x_2)}\) and there exists a selection \(p\) of \(P\) such that \(f_2 (p (\cdot), \cdot) \in M_2 (X_2, X_2).\) Let \(x_2^* \in F_{f_2 (p (\cdot), \cdot)}\). Then \((p (x_2^*), x_2^*) \in F_f.\)

**Problem 5.4.** The problem is to use the above heuristic to give fixed point results on cartesian product.

References: [2], [24], [35], [36], [37], [17], [22], [31], [6], [14], [21], [39], [40], [4], [9].

References


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