ASYMPTOTIC PROPERTIES OF SOLUTIONS TO DISCRETE STURM-LIOUVILLE MONOTONE TYPE EQUATIONS

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ABSTRACT. We investigate the discrete equations of the form
\[ \Delta(r_n \Delta x_n) = a_n f(x_{\sigma(n)}) + b_n. \]
Using the Knaster-Tarski fixed point theorem, we study solutions with prescribed asymptotic behaviour. Our technique allows us to control the degree of approximation. In particular, we present the results concerning harmonic and geometric approximations of solutions.

1. Introduction

Let \( \mathbb{N}, \mathbb{R} \) denote the set of positive integers and all real numbers, respectively. In this paper we assume \( a, b : \mathbb{N} \to \mathbb{R}, \ r : \mathbb{N} \to (0, \infty), \ f : \mathbb{R} \to \mathbb{R}, \ \sigma : \mathbb{N} \to \mathbb{N}, \ \lim_{n \to \infty} \sigma(n) = \infty, \) and consider the second order discrete equation with quasidifferences of the form
\[ \Delta(r_n \Delta x_n) = a_n f(x_{\sigma(n)}) + b_n. \]
By a solution of equation (E), we mean a sequence \( x \) which satisfies the equality (E) for all large \( n \). We say that equation (E) is of monotone type if one of the following conditions is satisfied:
(a) \( f \) is nondecreasing and \( a_n \geq 0 \) for any \( n \),
(b) \( f \) is nonincreasing and \( a_n \leq 0 \) for any \( n \).
Prescribed asymptotic behaviour to difference equations was discussed, for example, in [1–8,10,15].

Let us note that the following equation
\[ \Delta(r_n \Delta x_n) = a_n x_{n+1} \]  
(1)
is a particular case of equation (E). The equation (1) is known as Sturm-Liouville difference equation and has many applications in mathematical physics, matrix theory, control theory, and discrete variational theory. See, for example, the monographs [1–3]. The results devoted to the asymptotic behaviour to equation (1) can be found in [4–7,9,12–14].

In this paper, we investigate the existence of solutions to equation (E) with prescribed asymptotic behaviour. Such studies are usually based on applying the Schauder or Darboux type fixed point theorems. Here, we used the Knaster-Tarski theorem. The use of this theorem allows us to replace the conditions of continuity of the function \( f \) by the conditions of monotonicity type of \( f \).

We establish conditions under which for a given solution \( y \) of the equation
\[ \Delta(r_n \Delta y_n) = b_n \]there exists a solution \( x \) of (E) such that
\[ x_n = y_n + o(u_n), \]
where \( u \) is a given positive, nonincreasing sequence. Taking different sequences \( u \), we may control the degree of approximation. For example, if \( u_n = n^s \) for some fixed \( s \in (-\infty, 0) \) we have harmonic approximation. If \( u_n = \alpha^n \) for some fixed \( \alpha \in (0, 1) \), then we get the geometric approximation.

2. Preliminaries

We say that a sequence \( y \in \mathbb{R}^N \) is eventually \( f \)-bounded if there exist an index \( q \) and a positive constant \( \alpha \) such that the function \( f \) is bounded on the set
\[ \bigcup_{n=q}^{\infty} [y_n - \alpha, y_n + \alpha]. \]

Below we present some lemmas that will be used to prove the main theorem. The easy proof of the first of them we leave to the reader.

**Lemma 2.1.** If \( n \in \mathbb{N} \) and
\[ \sum_{k=1}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} |x_i| < \infty, \]
then
\[ \Delta \left( r_n \Delta \left( \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} x_i \right) \right) = x_n. \]

**Lemma 2.2.** If \( x : \mathbb{N} \rightarrow \mathbb{R}, \ u : \mathbb{N} \rightarrow (0, \infty), \ \Delta u \leq 0, \) and
\[ \sum_{k=1}^{\infty} \frac{1}{r_k u_k} \sum_{i=k}^{\infty} |x_i| < \infty, \]
then
\[ \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} x_i = o(u_n). \]
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Proof. Define a sequence \( z \) by

\[
z_n = \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} x_i.
\]

Then

\[
\frac{|z_n|}{u_n} \leq \frac{1}{u_n} \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} |x_i| = \sum_{k=n}^{\infty} \frac{1}{u_n r_k} \sum_{i=k}^{\infty} |x_i| \leq \sum_{k=n}^{\infty} \frac{1}{u_k r_k} \sum_{i=k}^{\infty} |x_i| = o(1).
\]

Hence \( z_n = o(u_n) \). \( \square \)

We will use the following consequence of the Knaster-Tarski fixed point theorem.

**Lemma 2.3 ([Lemma 4.9]).** Let \( y, \rho \in \mathbb{R}^N \) and let \( S \) denote the set

\[
\{ x \in \mathbb{R}^N : |x - y| \leq |\rho| \}
\]

with natural order defined by: \( x \leq z \) if \( x_n \leq z_n \) for any \( n \in \mathbb{N} \). Then every nondecreasing map \( T: S \to S \) has a fixed point.

We will use the convention \( \sum_{k=p}^{k} = 0 \) whenever \( k < p \).

**Remark 1.** A sequence \( y \) is a solution of the equation \( \Delta(r_n \Delta y_n) = b_n \) if and only if there exist real constants \( c, d \) such that

\[
y_n = \sum_{j=1}^{n-1} \frac{1}{r_j} \sum_{i=1}^{j-1} b_i + c \sum_{j=1}^{n-1} \frac{1}{r_j} + d
\]

for any \( n \).

3. Main results

The main result of the paper is the following theorem.

**Theorem 3.1.** Assume \((E)\) is of monotone type,

\[
u : \mathbb{N} \to (0, \infty), \quad \Delta u \leq 0, \quad \sum_{k=1}^{\infty} \frac{1}{r_k u_k} \sum_{i=k}^{\infty} |a_i| < \infty, \quad (2)
\]

and \( y \) is an eventually \( f \)-bounded solution of the equation \( \Delta(r_n \Delta y_n) = b_n \). Then there exists a solution \( x \) of \((E)\) such that \( x_n = y_n + o(u_n) \).
Proof. Choose an index \( q \) and positive constants \( \alpha \) and \( L \) such that \( |f(t)| \leq L \) for any \( t \) in the set

\[
U = \bigcup_{n=q}^{\infty} [y_n - \alpha, y_n + \alpha].
\]

Let

\[
Y = \{ x \in \mathbb{R}^N : |y - x| \leq \alpha \}.
\]

For \( n \in \mathbb{N} \) let

\[
\rho_n = L \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} |a_i|.
\]

Using Lemma 2.2 we get

\[
\rho_n = o(u_n) = o(1). \quad (3)
\]

Therefore, there exists an index \( p \) such that

\[
\rho_n \leq \alpha \quad \text{and} \quad \sigma(n) \geq q
\]

for \( n \geq p \). Let

\[
X = \{ x \in \mathbb{R}^N : |x - y| \leq \rho \quad \text{and} \quad x_n = y_n \quad \text{for} \quad n < p \}.
\]

Note that \( X \subset Y \). If \( x \in Y \) and \( n \geq p \), then \( |f(x_{\sigma(n)})| \leq L \). Define an operator

\[
T : Y \to \mathbb{R}^N \quad \text{by} \quad T(x)(n) = \begin{cases} y_n & \text{for} \quad n < p, \\ y_n + \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} a_i f(x_{\sigma(i)}) & \text{for} \quad n \geq p. \end{cases}
\]

If \( x \in X \), then for \( n \geq p \) we have

\[
|T(x)(n) - y_n| \leq \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} |a_i f(x_{\sigma(i)})| \leq L \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} |a_i| = \rho_n.
\]

Therefore \( T(X) \subset X \). Let \( x, z \in X, x \leq z \). For \( n \geq p \), using the monotonicity of \( (E) \), we have

\[
T(x)(n) = y_n + \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} a_i f(x_{\sigma(i)}) \leq y_n + \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} a_i f(z_{\sigma(i)}) = T(z)(n).
\]

Hence \( T(x) \leq T(z) \).
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Therefore the operator $T : X \rightarrow X$ is nondecreasing. By Lemma 2.3 there exists a point $x \in X$ such that $T(x) = x$.

Then for $n \geq p$ we have

$$x_n = y_n + \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} a_i f(x_{\sigma(i)}).$$

By Lemma 2.1 $x$ is a solution of (E). Since $x \in X$ and $\rho_n = o(u_n)$, we have $x_n = y_n + o(u_n)$. □

Theorem 3.1 has many interesting consequences. Below we present some of them.

**Corollary 3.2.** Assume (E) is of monotone type and (2) holds. Then for any bounded solution $y$ of the equation $\Delta(r_n \Delta y_n) = b_n$ there exists a solution $x$ of (E) such that $x_n = y_n + o(u_n)$.

**Proof.** The function $f$, as a monotone function, is bounded on any bounded subset of $\mathbb{R}$. Hence the assertion is a consequence of Theorem 3.1. □

Obviously, if the function $f$ is bounded, then any sequence $y \in \mathbb{R}^N$ is eventually $f$-bounded. Hence we get the following corollaries.

**Corollary 3.3.** Assume (E) is of monotone type, $f$ is bounded, and (2) holds. Then for any solution $y$ of the equation $\Delta(r_n \Delta y_n) = b_n$ there exists a solution $x$ of (E) such that $x_n = y_n + o(u_n)$.

**Corollary 3.4.** Assume (E) is of monotone type, (2) holds, $y$ is a solution of the equation $\Delta(r_n \Delta y_n) = b_n$, and one of the following conditions are satisfied:

(a) $\lim_{t \to \infty} f(t) < \infty$, $\liminf_{n \to \infty} y_n > -\infty$,
(b) $\lim_{t \to -\infty} f(t) > -\infty$, $\limsup_{n \to \infty} y_n < \infty$.

Then there exists a solution $x$ of (E) such that $x_n = y_n + o(u_n)$.

**Corollary 3.5.** Assume (E) is of monotone type, (2) holds,

$$\sum_{k=1}^{\infty} \frac{1}{r_k} < \infty,$$

and one of the following conditions is satisfied:

(a) $\lim_{t \to \infty} f(t) < \infty$, $b_n \geq 0$ for any $n \in \mathbb{N}$,
(b) $\lim_{t \to -\infty} f(t) > -\infty$, $b_n \leq 0$ for any $n \in \mathbb{N}$.

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Then for any solution \( y \) of the equation \( \Delta(r_n \Delta y_n) = b_n \) there exists a solution \( x \) of (E) such that \( x_n = y_n + o(u_n) \).

The next corollary concerns the existence of asymptotically periodic solutions.

**Corollary 3.6.** Assume (E) is of monotone type, (2) holds, \( m \in \mathbb{N} \), and \( y \) is an \( m \)-periodic solution \( y \) of the equation \( \Delta(r_n \Delta y_n) = b_n \). Then for any real constant \( d \) there exists an asymptotically \( m \)-periodic solution \( x \) of (E) such that

\[
x_n = y_n + d + o(u_n).
\]

**Proof.** Since \( y + d \) is a bounded solution of the equation \( \Delta(r_n \Delta y_n) = b_n \), the assertion follows from Corollary 3.2. \( \square \)

The below example shows that the equation \( \Delta(r_n \Delta y_n) = b_n \) can have 3-periodic solution.

**Example.** Assume \( b \) is a sequence such that

\[
b_n = \begin{cases} 
-3r_n & \text{for } n = 3k, \quad k \in \mathbb{N}, \\
-3r_{n+1} & \text{for } n = 3k + 1, \quad k \in \mathbb{N}, \\
3r_n + 3r_{n+1} & \text{for } n = 3k + 2, \quad k \in \mathbb{N}.
\end{cases}
\]

It is easy to see that the 3-periodic sequence \( y = (5, 5, 2, 5, 5, 2, \ldots) \) is a solution of the equation \( \Delta(r_n \Delta y_n) = b_n \).

**Remark 2.** If \( m \in \mathbb{N} \) and \( b \in \mathbb{R}^\mathbb{N} \), then, by \( \mathbb{I} \) Remark 2.4, the existence of an \( m \)-periodic solution of the equation \( \Delta(r_n \Delta y_n) = b_n \) is equivalent to the existence of a real constant \( c \) such that the sequence

\[
\alpha_n = \frac{c}{r_n} + \sum_{i=1}^{n-1} \frac{b_i}{r_n}
\]

is \( m \)-periodic and \( \alpha_1 + \alpha_2 + \cdots + \alpha_m = 0 \).

We omit the proof of the following easy lemma.

**Lemma 3.7.** If \( g, w : \mathbb{N} \to [0, \infty) \), \( \sum_{j=1}^{\infty} g_j < \infty \), and \( n \in \mathbb{N} \), then

\[
\sum_{k=n}^{\infty} w_k \sum_{j=k}^{\infty} g_j = \sum_{k=n}^{\infty} g_k \sum_{j=n}^{\infty} w_j.
\]
Now we present a result concerning harmonic approximation.

**Corollary 3.8.** Assume \((E)\) is of monotone type, \(y\) is an eventually \(f\)-bounded solution of the equation \(\Delta(r_n \Delta y_n) = b_n\),

\[
s \in (-\infty, 0], \quad \tau \in [s, \infty), \quad r_n^{-1} = O(n^\tau),
\]

and

\[
\sum_{n=1}^{\infty} n^{1+\tau-s}|a_n| < \infty.
\] (4)

Then there exists a solution \(x\) of \((E)\) such that \(x_n = y_n + o(n^s)\).

**Proof.** Let \(u_n = n^s\) for any \(n\). Choose a positive constant \(M\) such that \(r_k^{-1} \leq Mk^\tau\) for any \(k\). Using Lemma 3.7 we get

\[
\sum_{k=1}^{\infty} \frac{1}{r_k u_k} \sum_{i=k}^{\infty} |a_i| \leq M \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} k^{\tau-s} |a_i|
\]

\[
\leq M \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} i^{\tau-s} |a_i|
\]

\[= M \sum_{k=1}^{\infty} k^{1+\tau-s} |a_k| < \infty.
\]

Now, using Theorem 3.1, we obtain the result. \(\square\)

The following result is devoted to geometric approximation.

**Corollary 3.9.** Assume \((E)\) is of monotone type, \(y\) is an eventually \(f\)-bounded solution of the equation \(\Delta(r_n \Delta y_n) = b_n\),

\[
\tau \in \mathbb{R}, \quad r_n^{-1} = O(n^\tau), \quad \text{and} \quad \limsup_{n \to \infty} \sqrt[\tau]{|a_n|} < \beta < 1.
\] (5)

Then there exists a solution \(x\) of \((E)\) such that \(x_n = y_n + o(n^s)\).

**Proof.** Let \(u\) be a sequence defined by \(u_n = \beta^n\). Choose a positive constant \(M\) such that \(r_n^{-1} \leq Mn^\tau\) for any \(n\). For \(k \in \mathbb{N}\) we have

\[
\sum_{i=1}^{k} \frac{1}{r_i u_i} \leq M \sum_{i=1}^{k} \frac{i^{\tau}}{\beta^i} \leq M \sum_{i=1}^{k} \frac{k^{\tau}}{\beta^k} = M \frac{k^{\tau+1}}{\beta^k}.
\]

Hence, using Lemma 3.7, we get

\[
\sum_{k=1}^{\infty} \frac{1}{r_k u_k} \sum_{i=k}^{\infty} |a_i| = \sum_{k=1}^{\infty} |a_k| \sum_{i=1}^{k} \frac{1}{r_i u_i} \leq M \sum_{k=1}^{\infty} \frac{|a_k| k^{\tau+1}}{\beta^k}.
\]
Moreover,
\[
\limsup_{k \to \infty} \sqrt[k]{k^{\frac{1}{k+1}}} = \limsup_{k \to \infty} \frac{\sqrt[k]{a_k}}{\beta} < 1.
\]
Therefore
\[
\sum_{k=1}^{\infty} \frac{1}{r_k u_k} \sum_{i=k}^{\infty} |a_i| < \infty,
\]
and the assertion is a consequence of Theorem 3.1. □

In Theorem 3.1 we do not impose any restrictions on the sequence \( b \). Assuming \( b \) is “sufficiently small” we can obtain simpler asymptotic behaviour of solutions to equation (E) by replacing solutions \( y \) of the equation \( \Delta(r_n \Delta y_n) = b_n \) by solutions \( y \) of the equation \( \Delta(r_n \Delta y_n) = 0 \).

**Theorem 3.10.** Assume (E) is of monotone type,
\[
u : \mathbb{N} \to (0, \infty), \quad \Delta u \leq 0, \quad \sum_{k=1}^{\infty} \frac{1}{r_k u_k} \sum_{i=k}^{\infty} (|a_i| + |b_i|) < \infty, \tag{6}
\]
and \( y \) is an eventually \( f \)-bounded solution of the equation \( \Delta(r_n \Delta y_n) = 0 \). Then there exists a solution \( x \) of (E) such that \( x_n = y_n + o(u_n) \).

**Proof.** Choose an index \( q \) and a positive number \( \alpha \) such that \( f \) is bounded on \( U = \bigcup_{n=q}^{\infty} [y_n - \alpha, y_n + \alpha] \).

Choose a number \( \alpha' \in (0, \alpha) \) and let \( \beta = \alpha - \alpha' \). Define sequences \( v, y' \) by
\[
v_n = \sum_{k=n}^{\infty} \frac{1}{r_k u_k} \sum_{i=k}^{\infty} b_i, \quad y'_n = y_n + v_n.
\]
Using Lemma 2.2 we get \( v_n = o(u_n) \). Hence there exists an index \( q' \geq q \) such that \( |v_n| \leq \beta \) for any \( n \geq q' \). Let
\[
U' = \bigcup_{n=q'}^{\infty} [y'_n - \alpha, y'_n + \alpha'].
\]
If \( t \in U' \), then there exists an index \( k \geq q' \) such that \( |t - y'_k| \leq \alpha' \). Then
\[
|t - y_k| = |t - y'_k + y'_k - y_k| \leq |t - y'_k| + |y'_k - y_k| \leq \alpha' + |v_k| \leq \alpha' + \beta = \alpha.
\]
Hence \( U' \subset U \). Therefore \( f \) is bounded on \( U' \).

Using Lemma 2.1 we get \( \Delta(r_n \Delta v_n) = b_n \). Thus
\[
\Delta(r_n \Delta y'_n) = \Delta(r_n \Delta y_n) + \Delta(r_n \Delta v_n) = 0 + b_n = b_n.
\]
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By Theorem 3.1 there exists a solution $x$ of (E) such that $x_n = y'_n + o(u_n)$. Then
$$x_n = y_n + v_n + o(u_n) = y_n + o(u_n).$$

\[ \square \]

**Corollary 3.11.** Assume (E) is of monotone type and (6). Then for any bounded solution $y$ of the equation $\Delta(r_n y_n) = 0$ there exists a solution $x$ of (E) such that $x_n = y_n + o(u_n)$.

**Corollary 3.12.** Assume (E) is of monotone type and (6). Then for any real constant $c$ there exists a solution $x$ of (E) such that $x_n = c + o(u_n)$.

**Corollary 3.13.** Assume (E) is of monotone type, (6), and
$$\sum_{k=1}^{\infty} \frac{1}{r_k} < \infty.$$ Then for any real constants $c, d$ there exists a solution $x$ of (E) such that
$$x_n = c + d \sum_{k=1}^{n-1} \frac{1}{r_k} + o(u_n).$$

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